

On Raw Coding of Chaotic Dynamics

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Abstract

We study raw coding of trajectories of a chaotic dynamical system by sequences of elements from a finite alphabet and show that there is a fundamental constraint on differences between codes corresponding to different trajectories of the dynamical system.

1 Introduction

By a *raw coding* of a chaotic discrete time dynamical system (T, X, \mathcal{B}, μ) (defined by a measurable map T of a measurable space (X, \mathcal{B}, μ) into itself with a σ -algebra of measurable sets \mathcal{B} and a T -invariant probability measure μ), we mean a representation of trajectories $\{T^t x\}_{t \in \mathbb{Z}_+}$ of this system as sequences of elements from a finite alphabet \mathcal{A} . In other words, one defines a mapping $\Xi : X \rightarrow \mathcal{A}$, which takes points of the original phase space X of the dynamical system to elements of the alphabet \mathcal{A} . This mapping defines a partition $\xi := \{X_1, X_2, \dots, X_M\}$ of the phase space X into disjoint measurable subsets $X_i \in \mathcal{B}$, $i \in \{1, 2, \dots, M\}$, with a subsequent encoding of a trajectory by a sequence of numbers corresponding to elements of the partition which the trajectory successively visits; i.e., $\Xi(x) := i$ if $x \in X_i$. In some cases (if there exists a finite Markov partition; see, e.g., [1, 2]), information about raw encoded trajectories allows one to completely reconstruct all topological characteristics of the system under consideration. Note however that, even if the existence of a finite Markov partition is proved rigorously, its practicable application is rather questionable due to the high instability of the chaotic map T . Moreover, it might happen that an approximate

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Markov partition has even worse statistical characteristics (i.e., more differing from the characteristics of the original system) than an “arbitrary” one [3].

The aim of the present paper is to demonstrate that, under a rather weak assumption about a deterministic dynamical system (weak mixing condition), there is a fundamental constraint on elementwise differences between codes corresponding to different trajectories. Using standard definitions and constructions of ergodic theory, the main of which will be described in Section 2 and whose detailed analysis can be found, e.g., in [2, 4], this statement can be formulated as follows.

Theorem. Let a dynamical system (T, X, \mathcal{B}, μ) satisfy the weak mixing property. Consider a finite measurable partition $\xi := \{X_1, X_2, \dots, X_M\}$ with $\prod_{i=1}^M \mu(X_i) > 0$. Then, for any positive integers $N, L < \infty$ and for almost every (with respect to the direct product measure μ^N) collection $\bar{x} := \{x_1, x_2, \dots, x_N\} \in X^N$, there exists a time moment $t_0 \geq 0$ such that $\Xi(T^{t_0+t}x_i) = \Xi(T^{t_0+t}x_j)$ for all $t \in \{1, \dots, L\}$, $i, j \in \{1, \dots, N\}$.

In other words, all of these N codes of different trajectories simultaneously contain arbitrarily long subsequences coinciding both in space and time. In Section 3, after the proof of this result, we discuss possible weakenings of the assumptions under which it holds.

From the point of view of numerous applications, let us mention connections of this problem to the analysis of properties of random number generators. Let $\{x_i^t\}_{t \in \mathbb{Z}_+}$, $i \in \{1, \dots, N\}$, be N different sequences of pseudorandom numbers obtained from the same random number generator. We assume that those pseudorandom numbers belong to a finite alphabet \mathcal{A} . It is desirable that, despite a proximity (ideally, coincidence) of statistical properties of these realizations, their pointwise differences should be as large as possible. The theorem shows that the pointwise differences between different realizations are fundamentally constrained. Although this claim is somewhat unexpected, it completely agrees with well-known properties of Bernoulli or, more generally, Markov random sequences. Indeed, consider a Bernoulli random sequence over an alphabet with two elements $\{0, 1\}$, taken with probabilities p and $q = 1 - p$, respectively. Consider an arbitrary finite sequence \bar{a} over this alphabet, with $K \leq L$ elements 0 and $L - K$ elements 1. For any given positive integer t_0 , the probability that a realization of this Bernoulli random sequence coincides with \bar{a} on the time interval from t_0 to $t_0 + L - 1$ is $p^K q^{L-K} > 0$; it does not depend on an initial moment t_0 . Thus,

for an arbitrary finite number of realizations of this Bernoulli sequence, the claim of the theorem holds true with probability one. One can similarly prove this statement for a Markov random chain with a finite number of states $A = \{1, \dots, M\}$ and a transition probability matrix π satisfying the assumption $\pi^\varkappa > 0$ for some $\varkappa \in \mathbb{Z}_+$ (this is a direct analog of the weak mixing property for a Markov chain with a finite number of states).

It is worth to note connections of this problem to the Ulam finite Markov chain approximation scheme and to the so-called Bowen specification property (see, e.g. [4, 1]) for chaotic dynamical systems.

2 Necessary definitions and constructions

Here we give a short description of standard definitions and constructions from ergodic theory which we need to prove the theorem.

Recall that a measure μ is *T-invariant* if and only if

$$\int \varphi \circ T d\mu = \int \varphi d\mu$$

for any μ -integrable function $\varphi: X \rightarrow \mathbb{R}^1$.

A measurable function $\varphi: X \rightarrow \mathbb{R}^1$ is called *invariant* with respect to a dynamical system (T, X, \mathcal{B}, μ) (or simply *T-invariant*) if $\varphi = \varphi \circ T$ almost everywhere with respect to the measure μ .

A dynamical system (T, X, \mathcal{B}, μ) is *ergodic* if each *T*-invariant function is a constant μ -a.e.

A dynamical system (T, X, \mathcal{B}, μ) is *weakly mixing* if

$$\frac{1}{n} \sum_{k=0}^{n-1} |\mu(T^{-k}A \cap B) - \mu(A)\mu(B)| \xrightarrow{n \rightarrow \infty} 0, \quad \forall A, B \in \mathcal{B}.$$

A *direct product* of two dynamical systems $(T', X', \mathcal{B}', \mu')$ and $(T'', X'', \mathcal{B}'', \mu'')$ is a new dynamical system $(T' \otimes T'', X' \otimes X'', \mathcal{B}' \otimes \mathcal{B}'', \mu' \otimes \mu'')$, where the map $T' \otimes T'': X' \otimes X'' \rightarrow X' \otimes X''$ is defined by the relation

$$T' \otimes T''(x', x'') := (T'x', T''x''),$$

while all other objects are standard direct product of spaces, σ -algebras, and measures, respectively.

By A^N we denote the *N*th power of the set *A*, i.e., the direct product of *N* identical sets $A \in \mathcal{B}$, and by $(T^{\otimes N}, X^N, \mathcal{B}^N, \mu^N)$ the direct product of *N* identical dynamical systems (T, X, \mathcal{B}, μ) .

A collection $\xi := \{X_1, X_2, \dots, X_M\}$ is called a *measurable partition* of a measurable space (X, \mathcal{B}, μ) if $X_i \in \mathcal{B} \forall i$, $X_i \cap X_j = \emptyset \forall i \neq j$, and $\bigcup_i X_i = X$.

3 Proof of the theorem

The proof of this result consists of two steps. First we show that a sufficient condition for the claim of the theorem is the ergodicity of the direct product of N copies of the original dynamical system. Then we demonstrate that the latter property follows from the conditions of the theorem.

Denote $\xi^{(0)} := \xi$ and inductively define a sequence of measurable partitions $\xi^{(n)}$ with $n \in \mathbb{Z}_+$ by the following relation:

$$\xi^{(n+1)} := \xi^{(n)} \cap T^{-1}\xi^{(n)}.$$

It is immediate to check that the constructed collections of sets $\xi^{(n)}$ are measurable partitions of the space (X, \mathcal{B}, μ) for any positive integer n . The partition $\xi^{(n)}$ is called the *nth refinement* of the partition ξ .

By means of these partitions, for each $n \in \mathbb{Z}_+$ we define a collection of sets

$$(\xi^{(n)})^N := \bigcup_{\eta \in \xi^{(n)}} \eta^N,$$

which are “toothed neighborhoods of the diagonal” in the direct product X^N (see Fig. 1).

Observe that the inclusion $\bar{x} := \{x_1, \dots, x_N\} \in \eta^N$ with $\eta \in \xi^{(n)}$ implies the inclusion

$$(T^{\otimes N})^t \bar{x} \in (T^{\otimes N})^t \eta^N \in (\xi^{(n-t)})^N$$

for all $t \in \{0, 1, \dots, n\}$; hence,

$$\Xi(T^t x_i) = \Xi(T^t x_j), \quad \forall i, j \in \{1, 2, \dots, N\}, \quad t \in \{0, 1, \dots, n\}.$$

Therefore, if we prove that, for μ^N -a.a. collection of points $\bar{x} := \{x_1, \dots, x_N\} \in X^N$, there exists a time moment $t_0 \geq 0$ such that

$$(T^{\otimes N})^{t_0} \bar{x} \in \eta^N \in (\xi^{(L)})^N,$$

then the claim of the theorem will follow.

Let a dynamical system $(\tau, Y, \mathcal{B}_Y, \nu)$ be ergodic. Then, for any pair of measurable sets $A, B \in \mathcal{B}_Y$ with $\nu(A)\nu(B) > 0$, there exists a positive integer $\varkappa = \varkappa(A, B) < \infty$ such that $\tau^\varkappa A \cap B \neq \emptyset$. Indeed, assume that

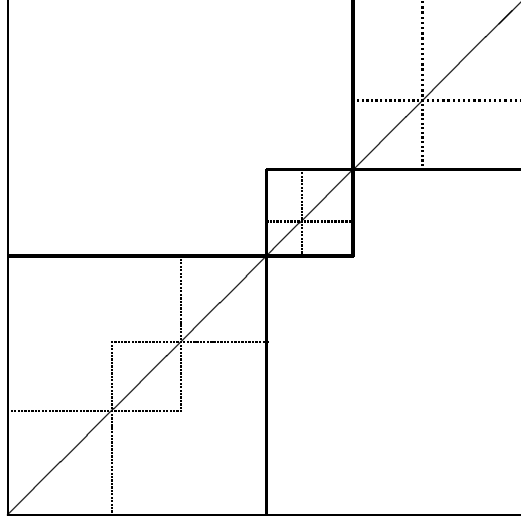


Figure 1: “Toothed neighborhoods of the diagonal” $[0, 1]^2$ created by collections of the sets $(\xi^{(n)})^2$ (the boundary is indicated by solid lines) and $(\xi^{(n+1)})^2$ (the boundary is indicated by dashed lines).

this is not true, i.e., $\tau^n A \cap B = \emptyset$ for any positive integer n . Consider a measurable set

$$A_\infty := \bigcup_{n \in \mathbb{Z}_+} \tau^n A.$$

Then $\nu(A_\infty) \geq \nu(A) > 0$ and $A_\infty \cap B = \emptyset$. Therefore, the indicator function of a measurable set A_∞ of positive ν -measure is τ -invariant but is not a constant a.e., which contradicts the ergodicity.

Thus, it suffices to show that the dynamical system $(T^{\otimes N}, X^N, \mathcal{B}^N, \mu^N)$ is ergodic. For that, we use the fact that the weak mixing property is preserved under the direct product of weakly mixing dynamical systems (see e.g. [2]). To complete the proof, it remains to note that weak mixing implies ergodicity. \square

It is interesting that one cannot weaken the conditions of the theorem by changing the weak mixing of the original dynamical system to the ergodicity. The problem is that the direct product of ergodic dynamical systems need not also be ergodic: consider a direct product of two identical irrational unit circle rotations.

On the other hand, the weak mixing condition is not necessary either.

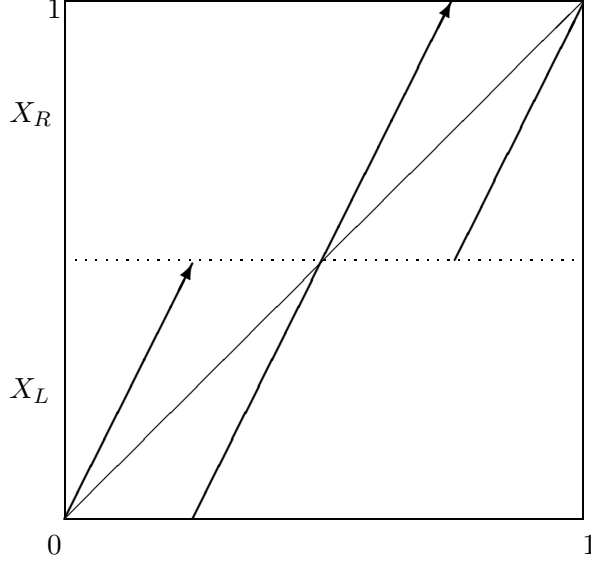


Figure 2: Counterexample to the necessity of the weak mixing property.

We shall show that, for some nonergodic dynamical systems, there exist partitions for which the claim of the theorem holds true. Indeed, let T be a mapping from the unit interval $X = [0, 1]$ into itself defined by the relation

$$Tx := \begin{cases} 2x & \text{if } 0 \leq x < 1/4, \\ 2x - 1/2 & \text{if } 1/4 \leq x < 3/4, \\ 2x - 1 & \text{otherwise.} \end{cases} \quad (1)$$

The graph of this mapping is shown in Fig. 2. It is immediate to check that the restriction of the map T to each of the half-intervals $X_L := [0, 1/2]$ (left) and $X_R := [1/2, 1]$ (right) with the normalized Lebesgue measure m is a weakly (and even strongly) mixing dynamical system. Thus, the Lebesgue invariant measure of the map T considered on the entire unit interval is not ergodic. Consider a partition ξ with $\prod_{i=1}^M \mu(X_i) > 0$ satisfying the assumption that $X_1 := [1/2 - 2^{-k}, 1/2 + 2^{-k})$ for some $k \in \mathbb{Z}_+$. Denote by ξ_L and ξ_R the restrictions of the partition ξ to the left and right half-intervals, respectively. The claim of the theorem is satisfied for each of the dynamical systems (T, X_L, \mathcal{B}, m) and (T, X_R, \mathcal{B}, m) equipped with the partitions ξ_L and ξ_R , respectively. Moreover, it turns out that, for the nonergodic dynamical system (T, X, \mathcal{B}, m) with the partition ξ , the claim of the theorem holds as well.

To check this, it is sufficient to show that, for any measurable set $A \subseteq [0, 1]^N$ of positive Lebesgue measure, there exists an integer t_0 such that the set $(T^{\otimes N})^{t_0} A$ has an intersection of positive Lebesgue measure with the “toothed neighborhood of the diagonal” in $[0, 1]^N$ generated by the N th power of the L th refinement of the partition ξ . Observe that, by the construction, each of these refinements has an element containing an open neighborhood of the point $1/2$. Denote by $B \in [0, 1]^N$ the N th power of this open neighborhood contained in an element of the partition $\xi^{(L)}$. We need to show that

$$(T^{\otimes N})^{t_0} A \cap B \neq \emptyset \quad (2)$$

for some positive integer t_0 .

By a *quadrant*, we shall call a direct product of $n \in \{0, 1, \dots, N\}$ copies of the interval X_L and $N - n$ copies of the interval X_R taken in an arbitrary order. Clearly, the union of all possible quadrants coincides with $[0, 1]^N$. On the other hand, each of the quadrants is invariant under the map $T^{\otimes N}$. Moreover, the restriction of the dynamical system $(T^{\otimes N}, X^N, \mathcal{B}^N, \mu^N)$ to each of the quadrants is again a direct product of mixing dynamical systems. Now there exists a quadrant whose intersection with the set A is of positive Lebesgue measure. Denote this quadrant by Δ and set $A_\Delta := A \cap \Delta$ and $B_\Delta := B \cap \Delta$. Observe that, by the construction, we have $m(A_\Delta)m(B_\Delta) > 0$. Therefore, using the same argument as in the proof of the theorem, we get

$$(T^{\otimes N})^{t_0} A_\Delta \cap B_\Delta \neq \emptyset$$

for some $t_0 > 0$, which implies (2).

Therefore, the claim of the theorem about the existence of arbitrarily long coinciding code segments takes place for this nonergodic dynamical system. In a sense, the element X_1 of the partition ξ plays a role of a “bridge” between ergodic components of the nonergodic dynamical system $(T, [0, 1], \mathcal{B}, m)$.

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